Low Peclet Number Mass or Heat Transfer In Two-Phase Particulate Systems

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High Peclet number convective heat or mass transfer from ensembles of drops, bubbles, or solid particles in slow viscous motion has recently been investigated by Waslo and Gal-Or (1971) and Yaron and Gal-Or (1971). It was shown that the Nusselt number is a function of the volume concentration of the dispersed phase Φ, and of the rate of internal circulation within the dispersed particles. The latter factor was evaluated by means of the viscosity parameter β, which was defined as the ratio of the continuous phase viscosity μ^c to the sum of the dispersed phase viscosity μ^d and the interfacial retardation viscosity γ, due to adventitious surfactant impurities adsorbed at the particles' interfaces [for significance and derivations of γ see Gal-Or and Waslo (1968)]. It was found that at high β values, that is, for high rates of internal circulation, the Nusselt number was proportional to $Pe^{1/2}$, whereas for noncirculating particles $Nu \propto Pe^{1/3}$.

The applicability of the above results is restricted to high Peclet numbers ($Pe \sim 10^5$), that is, to liquid continuous phases with very high Prandtl or Schmidt numbers. In many commonly encountered applications one is concerned with assemblages of particles, in slow viscous motion, (that is, at $Re << 1$) within low Prandtl or Schmidt number media, for example, gases. In such situations the continuous phase Peclet number would be of the order of 1. In the range of $Pe < 1$ the analyses of Friedlander (1957) and Bowman et al. (1961) are available for heat or mass transfer from single solid spheres. Their results are extended here to unbounded assemblages of circulating drops or noncirculating drops or solid particles with the aid of the statistical spherical cell model. The basic assumptions and concepts underlying this model have been presented by Gal-Or (1970) and will not be repeated here.

The purpose of the present communication is, therefore, to investigate the dependence of mass or heat transfer in the continuous phase of particulate systems at low Peclet numbers upon the volume concentration Φ and the rate of internal circulation within the dispersed phase.

THEORY

Following Bowman et al. (1961) and customarily neglecting tangential diffusion terms, we put the conservation of mass equation* in the continuous phase around a typical particle of a two-phase binary system, in the dimensionless form

$$\frac{d}{d\theta}\left[\int_0^{\Psi_\delta} C d\Psi\right] = \frac{2}{Pe}\left(\frac{\partial C}{\partial \eta}\right)_{\eta=1} \sin\theta \qquad (1)$$

where

$$C = \frac{c - c_\delta}{c_i - c_\delta} \qquad (2)$$

$$Pe = \frac{2aU_s}{D} \qquad (3)$$

$$\eta = \frac{r}{a} \qquad (4)$$

$$\Psi = \frac{\psi}{U_s a^2} \qquad (5)$$

We assume a dimensionless concentration profile of the polynomial form

$$C = \sum_j A_j \eta^j \qquad (6)$$

which must satisfy the conditions

$$\left.\begin{aligned} &\text{For } \eta = 1: \quad C = 1, \\ &\qquad\qquad \frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial C}{\partial \eta}\right) = 0 \\ &\text{For } \eta = \delta: \quad C = \frac{\partial C}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\eta^2 \frac{\partial C}{\partial \eta}\right) = 0 \end{aligned}\right\} \qquad (7)$$

Accordingly

$$C = \frac{1}{2(\delta - 1)^3}\left[3\delta(2 - \delta) + \delta(2\delta - 3)\frac{1}{\eta} - 3\eta + \eta^2\right] \qquad (8)$$

It also follows from boundary condition (7) that

$$\int_0^{\Psi_\delta} C d\Psi = -\int_1^0 \Psi dC \qquad (9)$$

We now introduce expression (9) into Equation (1) together with the concentration profile (8) and the Gal-Or and Waslo (1968) dimensionless stream function

$$\Psi = \frac{3}{4}\frac{1}{W}\left(\frac{1}{\eta} - W + Y\eta^2 - \Phi^{5/3}\eta^4\right)\sin^2\theta \qquad (10)$$

where

$$W = 3\left(1 + \frac{2}{3}\beta\right) + 2\Phi^{5/3}(1 - \beta) \qquad (11)$$

$$Y = 2(1 + \beta) + 3\Phi^{5/3}\left(1 - \frac{2}{3}\beta\right) \qquad (12)$$

$$\beta = \frac{\mu^c}{\mu^d + \gamma} \qquad (13)$$

It should be emphasized that since expression (10) has been derived on the basis of the statistical spherical cell model, the values obtained in the present analysis for

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* The formulation for heat transfer is identical and is therefore omitted here.

mass or heat transfer with its aid will be typical expected values, averaged over the entire assemblage of particles, rather than actual values, liable to be measured around any given particle.

Integrating now Equation (1) and neglecting terms from order of magnitude considerations, we arrive at the following approximation for the concentration (or temperature) fields

$$C = \frac{B}{2}(32W - 2B)^{-3} \Big[3(32W - B)(3B - 32W) + (32W - B)(64W - 5B)\frac{1}{\eta} - 3\eta B^2 + \eta^2 B^2 \Big] \quad (14)$$

$$\eta \leq \Phi^{-1/3}$$

where

$$B = 9YPe(1 - \cos\theta) \quad (15)$$

Employing now the definition

$$Nu = \int_{\pi}^{0} \left(\frac{\partial C}{\partial \eta} \right)_{\eta=1} \sin\theta \, d\theta \quad (16)$$

we obtain

$$Nu = \int_{\pi}^{0} \frac{32W \sin\theta}{[32W - 9YPe(1 - \cos\theta)]} d\theta = \frac{32W}{9YPe} \ln \frac{1}{\left(1 - \frac{18Y}{32W} Pe\right)}$$

which, upon expansion in series, becomes

$$Nu = 2\left[1 + \frac{9}{32}\frac{Y}{W} Pe + \frac{27}{256}\left(\frac{Y}{W}\right)^2 Pe^2 + \ldots \right] \quad (17)$$

It can be readily shown that had one employed a second-power polynomial for C one would arrive at the result

$$Nu = 2\left[1 + \frac{3}{8}\frac{Y}{W} Pe + \frac{3}{16}\left(\frac{Y}{W}\right)^2 Pe^2 + \ldots \right] \quad (18)$$

It is interesting to note that in the limit of a single noncirculating particle, that is, for $\Phi \to 0$ and $\beta \to 0$, the ratio Y/W tends to 2/3, whereby Equation (17) reduces to the result of Bowman et al. (1961)

$$Nu = 2\left[1 + \frac{3}{16} Pe + \frac{9}{128} Pe^2 + \ldots \right] \quad (19)$$

and Equation (18) to that due to Friedlander (1957)

$$Nu = 2\left[1 + \frac{1}{4} Pe + \frac{1}{12} Pe^2 + \ldots \right] \quad (20)$$

Moreover, for the limiting case of a single strongly circulating particle ($\Phi \to 0$, $\beta \to \infty$, $Y/W \to 1$), Equations (17) and (18) tend, respectively, to

$$Nu = 2\left[1 + \frac{9}{32} Pe + \frac{27}{256} Pe^2 + \ldots \right] \quad (21)$$

and

$$Nu = 2\left[1 + \frac{3}{8} Pe + \frac{3}{16} Pe^2 + \ldots \right] \quad (22)$$

Finally, in the absence of convective contribution, namely, for $Pe \to 0$, Equations (17) and (18) reduce to $Nu = 2$, both for single particles and for assemblages of particles. The latter result is essentially different from that of Waslo and Gal-Or (1971), who have postulated for the stagnant case

$$Nu = 2/(1 - \Phi^{1/3})$$

TABLE 1. NUSSELT NUMBER VALUES FOR MASS OR HEAT TRANSFER FROM ENSEMBLES OR SINGLE PARTICLES AT LOW PECLET NUMBERS

	Negligible internal circulation ($\beta = 0.01$)		Strong internal circulation ($\beta = 100$)	
Φ	0	0.5	0	0.5
$Pe = 10^{-2}$	2.004	2.005	2.006	2.006
$Pe = 10^{-1}$	2.034	2.047	2.058	2.058
$Pe = 1$	2.471	2.596	2.769	2.769

CONCLUDING REMARKS

The effects of the volume fraction of the dispersed phase and degree of internal circulation upon the Nusselt number for mass or heat transfer at low Peclet numbers to or from particles in a concentrated particulate system are illustrated in Table 1.

The figures in Table 1 indicate that at very low Peclet numbers the volume content of the dispersed phase is a secondary factor in mass or heat transfer and begins to obtain significance for noncirculating particles at Peclet numbers of the order of unity. For these Peclet numbers transfer rates from strongly circulating particles may be some 10% higher than those from rigid ones. We, therefore, conclude that the effects of both Φ and β are much less prominent in this case than in the case of very large Peclet numbers (Waslo and Gal-Or, 1971; Yaron and Gal-Or, 1971).

NOTATION

A_j = coefficients in polynomial Equation (6)
a = radius of a spherical drop or solid particle
C = dimensionless concentration (or temperature) defined by Equation (2)
c = concentration of a binary system
D = mass diffusivity in a binary system
Nu = Nusselt number defined by Equation (16)
Pe = Peclet number defined by Equation (3)
r = radial distance from center of particle
U_s = Stokes' terminal velocity
W = dimensionless parameter defined by Equation (11)
Y = dimensionless parameter defined by Equation (12)

Greek Letters

β = dimensionless viscosity ratio defined by Equation (13)
γ = interfacial retardation viscosity defined in Gal-Or and Walso (1968)
δ = dimensionless radial distance to edge of thick boundary layer
η = dimensionless radial distance defined by Equation (4)
θ = polar angle measured from rear stagnation point
μ^c = viscosity of the continuous phase
μ^d = viscosity of the dispersed phase
Φ = volume fraction of the dispersed phase
Ψ = dimensionless stream function, defined by Equation (5)

ψ = stream function

Subscripts

i = at interface
δ = at edge of thick boundary layer

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A Modified Maximum Principle

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While the maximum principle of Pontryagin remains as the standard theoretical development for optimal control, the implementation of the theory to even modestly realistic systems remains a most difficult problem. This is because of the generation of two-point-boundary value (TPBV) equations which are, when combined with special instability features, almost impossible to solve except in trivial cases.

In this note we wish to point out a modified form of the maximum principle as detailed by Chuprun (1967) which removes the instability feature. As such, it opens the door further for the use of the maximum principle in optimal control problems.

THEORY

In one form of the maximum principle (Lapidus and Luus, 1967) the optimal control problem consists of a state equation and initial condition

$$\dot{x}(t) = f[x(t),\ u(t)] \tag{1}$$

$$x(0) = b \tag{2}$$

a final value index

$$S = cx(t_f) \tag{3}$$

and the object is to select the control $u(t)$ such that S is minimized while satisfying the constraints of (1) and (2). t_f is the final time of control and these equations can be visualized in scalar or vector-matrix form depending on the various dimensions.

The maximum principle proceeds by introduction of a scalar Hamiltonian function

$$H = f\lambda \tag{4}$$

with the resulting conditions for optimality

$$\begin{aligned} \dot{x} &= H_\lambda = f & x(0) &= b \\ \dot{\lambda} &= -H_x = -f_x\lambda & \lambda(t_f) &= -c \end{aligned} \tag{5}$$

$$H_u = 0 \tag{6}$$

Here $\lambda(t)$ is a new variable (the adjoint variable) which has been introduced and the condition of a stationary value of H (given by $\partial H/\partial u = H_u = 0$) yields the optimal control $u^0(t)$. In the regular problem (6) yields $u^0(t)$ in terms of $x(t)$ and $\lambda(t)$; in the singular problem (6) does not yield $u^0(t)$ and further information must be obtained. We here only consider the regular problem in which case the TPBV of (5) must be solved in order to remove the dependence of $x(t)$ and $\lambda(t)$ in $u^0(t)$.

In solving (5) it is frequently found that when $x(t)$ is stable in a positive time direction $\lambda(t)$ is unstable in the same direction (and vice-versa). It is important to realize however, that it is the direction of the adjoint equations as vectors in coordinate space and not their actual magnitude which is of importance. As indicated by Sagan (1969) $\lambda(t)$ need only be determined but for a multiplicative constant. This property will be used to advantage shortly.

Chuprun's method consists of replacing the adjoint variable $\lambda(t)$ with a new variable $\rho(t)$ such that

$$H = f\rho \tag{7}$$

and

$$\rho = -f_x\rho + e(\rho, t)\rho \tag{8}$$

where $e(\rho, t)$ is a scalar function (in general, nonlinear). He then proves that (8) and the second equation in (5) have identical directions for all times t and thus can be used interchangeably without altering the results of the standard maximum principle. Further he shows that as long as $e(\rho, t)$ is chosen carefully, the solution of (8) must remain on a sphere of constant radius, and (8) can be made stable or bounded in the positive time direction. In simple geometric terms, the introduction of $e(\rho, t)$ means that the normal velocity component of the representative point in phase space varies while the tangential component remains constant. It is also apparent that the

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